Nordhaus-Guddam Type Relations of Three Graph Coloring Parameters

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Abstract

Let G be a simple graph. A coloring of vertices of G is called (i) a 2-proper coloring if vertices at distance 2 receive distinct colors; (ii) an injective coloring if vertices possessing a common neighbor receive distinct colors; (iii) a square coloring if vertices at distance at most 2 receive distinct colors. In this paper, we study inequalities of Nordhaus-Guddam type for the 2-proper chromatic number, the injective chromatic number, and the square chromatic number.

Keywords: Nordhaus-Guddam type, 2-proper coloring, injective coloring, square coloring, chromatic number.

1 Introduction

Let $G = (V, E)$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* $|G|$ of G is the cardinality of $V(G)$. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v. The *maximum* and *minimum* degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of vertices adjacent to v. The *distance* $d_G(u, v)$ between two vertices u and

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v is the length of a shortest (u, v)-path. We abbreviate $d_G(u, v)$ to $d(u, v)$ when no ambiguity arises. A subset S of $V(G)$ is an *independent* set of G if $uv \notin E(G)$ for all vertices u and v in S. A subset W of $V(G)$ is a *clique* of G if $uv \in E(G)$ for all vertices u and v in W. A clique on n vertices is denoted by K_n. The *complement* $\overline{G}$ of G is the graph defined on the vertex set $V(G)$ of G such that an edge $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

Let k be a positive integer. A mapping $f : V(G) \to \{1, 2, \ldots, k\}$ is called a (proper) *k-coloring* of G if $f(u) \neq f(v)$ whenever $uv \in E(G)$. The *chromatic number* $\chi(G)$ of G is the minimum number k such that G has a k-coloring. The following is a well-known theorem of Nordhaus and Guddam [7].

Theorem 1 *If G is a graph of order n, then*

1. $2\sqrt{n} \leqslant \chi(G) + \chi(\overline{G}) \leqslant n + 1$.

2. $n \leqslant \chi(G)\chi(\overline{G}) \leqslant (n + 1)^2/4$.

Inequalities involving the sum or product of a parameter applied to a graph and its complement are commonly known as Nordhaus-Guddam type relations. The reader is referred to Aouchiche and Hansen [1] for a recent survey.

A mapping $f : V(G) \to \{1, 2, \ldots, k\}$ is called

- a *2-proper k-coloring* of G if $f(u) \neq f(v)$ whenever $d(u, v) = 2$;
- an *injective k-coloring* of G if $f(u) \neq f(v)$ whenever the u and v have a common neighbor;
- a *square k-coloring* of G if $f(u) \neq f(v)$ whenever $d(u, v) \leqslant 2$.

The minimum number k such that G has a 2-proper, an injective, or a square k-coloring is called the *2-proper, injective,* or *square chromatic number of G*. They are denoted by $\chi_2(G)$, $\chi_i(G)$, and $\chi_\Box(G)$, respectively. Let G^2 be the square graph of G obtained by adding a new edge between any pair of vertices that are distance 2 apart in G. Obviously, $\chi_\Box(G)$ is precisely $\chi(G^2)$.

The above graph colorings are closely related to a more general notion of graph labelings. Let p and q be two nonnegative integers. A k-$L(p, q)$-labeling of a graph G is a mapping $f : V(G) \to \{0, 1, \ldots, k\}$ such that $|f(u) - f(v)|$ is at least p if $d(u, v) = 1$

and at least q if $d(u, v) = 2$. The $L(p, q)$-labeling number $\lambda(G; p, q)$ of G is the least k such that G has a k-$L(p, q)$-labeling with $\max\{f(v) \mid v \in V(G)\} = k$. Obviously, an $L(1, 0)$-labeling of a graph G is a proper coloring of G and $\chi(G) = \lambda(G; 1, 0) + 1$; an $L(0, 1)$-labeling of a graph G is a 2-proper coloring of G and $\chi_2(G) = \lambda(G; 0, 1) + 1$; an $L(1, 1)$-labeling is a square coloring and $\chi_\square(G) = \lambda(G; 1, 1) + 1$. Note that, if G is triangle-free, then $\chi_i(G) = \chi_2(G)$. The reader is referred to Yeh [8] for a survey on $L(p, q)$-labelings of graphs. The injective coloring has been studied in [2–6].

In this paper, we study inequalities of Nordhaus-Guddam type for the 2-proper chromatic number, the injective chromatic number, and the square chromatic number. Graphs attaining extrema are also obtained.

2 2-proper chromatic numbers

For a given coloring of a graph, a *color class* consists of all vertices of a fixed color. Note that any color class of a 2-proper coloring consists of disjoint cliques. For $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r \geqslant 1$, let $K_{n_1,n_2,\ldots,n_r}$ denote the *complete r-partite graph* such that its vertex set has r disjoint parts with edges joining every pair of vertices belonging to different parts.

Lemma 2 *For $r \geqslant 2$, $\chi_2(K_{n_1,n_2,\ldots,n_r}) = n_1$.*

Proof. Let $\{V_1, V_2, \ldots, V_r\}$ denote the parts of $G = K_{n_1,n_2,\ldots,n_r}$ with $|V_i| = n_i$, $1 \leqslant i \leqslant r$. For each i, color the vertices of V_i with colors $1, 2, \ldots, n_i$ such that no pair of vertices receiving the same color to obtain a 2-proper coloring. Hence $\chi_2(G) \leqslant n_1$. Since any two vertices in V_1 are at distance 2, $\chi_2(G) \geqslant n_1$. ∎

Theorem 3 *For any graph G of order n,*

$$1 \leqslant (\chi_2(G)\chi_2(\overline{G}))^{1/2} \leqslant \frac{\chi_2(G) + \chi_2(\overline{G})}{2} \leqslant \frac{n+1}{2}.$$

Proof. It suffices to prove that $\chi_2(G) + \chi_2(\overline{G}) \leqslant n + 1$. Without loss of generality, we may suppose that $\chi_2(G) \geqslant \chi_2(\overline{G})$. If $\chi_2(G) \leqslant (n + 1)/2$, then $\chi_2(G) + \chi_2(\overline{G}) \leqslant n + 1$. Now assume that $\chi_2(G) > (n + 1)/2$.

Among all 2-proper colorings of G using $\chi_2(G)$ colors, let f be chosen with the maximum number of singleton color classes. Let $\{X_i\}_{i=1}^a$, $\{Y_j\}_{j=1}^b$, and $\{Z_k\}_{k=1}^c$ denote, respectively, the collections of color classes of f such that each X_i is a singleton,

each Y_j consists of a single clique of size at least two, and each Z_k consists of at least two disjoint cliques. Thus $\chi_2(G) = a + b + c > (n+1)/2$. First note that $a > 0$, for otherwise $n \geqslant 2b + 2c = 2\chi_2(G) > n + 1$.

Let $\mathcal{X} = \bigcup_{i=1}^{a} X_i$, $\mathcal{Y} = \bigcup_{j=1}^{b} Y_j$, and $\mathcal{Z} = \bigcup_{k=1}^{c} Z_k$. Then $\mathcal{X}$ must be an independent set, for otherwise we may re-color two adjacent vertices in $\mathcal{X}$ with the same color to obtain a 2-proper coloring of G using $\chi_2(G) - 1$ colors. The complement $\overline{G[Z_k]}$ of the subgraph $G[Z_k]$ induced by Z_k in G is a complete multipartite graph. By Lemma 2, $\chi_2(\overline{G[Z_k]}) \leqslant |Z_k| - 1$.

Now suppose that $b > 0$. There is a vertex u_1 of Y_1 that is non-adjacent to any vertex in $\mathcal{X}$. Otherwise, we could re-color each vertex y of Y_1 with color $f(x_{i_y})$, where $i_y = \min\{t \mid x_t \in N_G(y) \cap \mathcal{X}\}$, to obtain a 2-proper coloring of G with $\chi_2(G) - 1$ colors. Next, we move any vertex $v \in Y_1$ that is different from u_1 and adjacent to all vertices of Y_2 from Y_1 to Y_2. In view of the maximality of a, we are left with at least one $v_1 \in Y_1$ that is different from u_1 and non-adjacent to a certain vertex $u_2 \in Y_2$ if $b > 1$. We may repeat this process of moving vertices to the next color class until we obtain a sequence of vertices $u_1, v_1, u_2, v_2, \ldots, u_b, v_b$ such that $u_j, v_j \in Y_j$, and $u_j \neq v_j$ for $1 \leqslant j \leqslant b$ and $v_j u_{j+1} \in E(\overline{G})$ for $1 \leqslant j \leqslant b - 1$. Now, in $\overline{G}$, we color u_1 and the vertices in $\mathcal{X}$ with color 1, v_j and u_{j+1} with color $j + 1$ for $1 \leqslant j \leqslant b - 1$, and the vertices in $\mathcal{Y} \setminus \{u_1, v_1, \ldots, v_{b-1}, u_b\}$ with colors $b + 1, b + 2, \ldots, |\mathcal{Y}| - b + 1$ such that no pair of vertices receiving the same color. It follows that

$$\begin{aligned}
\chi_2(\overline{G}) &\leqslant \textstyle\sum_{k=1}^{c} \chi_2(\overline{G[Z_k]}) + |\mathcal{Y}| - b + 1 \\
&\leqslant |\mathcal{Z}| - c + |\mathcal{Y}| - b + 1 \\
&= n - a - b - c + 1 \\
&= n - \chi_2(G) + 1.
\end{aligned}$$

The above inequalities hold even if $b = 0$. Therefore, $\chi_2(G) + \chi_2(\overline{G}) \leqslant n + 1$. ∎

Let us consider the sharpness of inequalities in the above theorem. The lower bound is sharp since $\chi_2(K_n) = \chi_2(\overline{K_n}) = 1$. For the case of upper bound, we first construct an auxiliary graph H_k as follows. Let $k \geqslant 6$. The vertex set of H_k can be partitioned into an independent set $X = \{x_0, x_1, \ldots, x_{k-1}\}$ and a clique $Y = \{y_0, y_1, \ldots, y_{k-1}\}$ so that each x_i is joined to $y_i, y_{i+1}, \ldots, y_{i+\lfloor k/2 \rfloor}$ except $y_{i+\lfloor k/2 \rfloor - 1}$. Here indices are taken modulo k.

For $0 \leqslant i \neq j < k$, if $y_{i+\lfloor k/2 \rfloor - 1}$ and $y_{j+\lfloor k/2 \rfloor - 1}$ are not neighbors of both x_i and x_j, then x_i and x_j together have $2\lfloor k/2 \rfloor > k - 2$ edges joining Y. Hence, they must have a common neighbor and $d_{H_k}(x_i, x_j) = 2$. Suppose that x_i is adjacent to

$y_{j+\lfloor k/2\rfloor-1}$. Since $k \geqslant 6$, there are three possibilities: (i) $y_{i+\lfloor k/2\rfloor} = y_{j+\lfloor k/2\rfloor-1}$; (ii) $y_{i+\lfloor k/2\rfloor-2} = y_{j+\lfloor k/2\rfloor-1}$; (iii) $y_{i+t} = y_{j+\lfloor k/2\rfloor-1}$ for some $0 \leqslant t \leqslant \lfloor k/2\rfloor - 3$. Then x_i and x_j have a common neighbor z, where z is y_j for (i), y_{j+1} for (ii), and $y_{j+\lfloor k/2\rfloor}$ for (iii). Again, $d_{H_k}(x_i, x_j) = 2$.

The complement graph $\overline{H_k}$ can be isomorphically described as follows. Let $X = \{x_0, x_1, \ldots, x_{k-1}\}$ be a clique and $Y = \{y_0, y_1, \ldots, y_{k-1}\}$ be an independent set such that each y_i is joined to $x_i, x_{i+1}, \ldots, x_{i+\lceil k/2\rceil}$ except $x_{i+\lceil k/2\rceil-1}$. When k is even, $\overline{H_k}$ is isomorphic to H_k. When k is odd, any y_i and y_j, $i \neq j$, together have $2\lceil k/2\rceil = k+1$ edges joining X. It follows that $d_{\overline{H_k}}(y_i, y_j) = 2$ for $0 \leqslant i \neq j < k$.

In the second step, we construct a graph $H_{\rm od}$ of order $2k+1 \geqslant 13$ and a graph $H_{\rm ev}$ of order $2k+2 \geqslant 14$ as follows. We join a new vertex ∞ to all y_i's in H_k to obtain $H_{\rm od}$ and two new independent vertices ∞_1 and ∞_2 to all y_i's in H_k to obtain $H_{\rm ev}$. It is straightforward to see that $\chi_2(H_{\rm od})+\chi_2(\overline{H_{\rm od}}) = 2k+2$ and $\chi_2(H_{\rm ev})+\chi_2(\overline{H_{\rm ev}}) = 2k+3$.

3 Injective chromatic numbers

For the injective chromatic number $\chi_i(G)$ of a graph G, it is clear that $\Delta(G) \leqslant \chi_i(G) \leqslant |G|$. Note that if S is a color class of an injective k-coloring, then $\Delta(G[S]) \leqslant 1$.

Suppose G is a graph of order $n \leqslant 4$. It is routine to check that (i) $n \leqslant \chi_i(G) + \chi_i(\overline{G}) \leqslant 2n$ except $\chi_i(C_4) + \chi_i(\overline{C_4}) = 3$; (ii) $n \leqslant \chi_i(G)\chi_i(\overline{G}) \leqslant n^2$ except $G \in \{K_2$, $\overline{K_2}$, P_3, $\overline{P_3}$, C_4, $\overline{C_4}\}$. Here, P_n and C_n denote a path and a cycle on n vertices, respectively.

Lemma 4 *Suppose that the graph G has order $n \geqslant 5$. Then the following statements hold.*

(1) *If $\delta(G) \geqslant (n+1)/2$, then $\chi_i(G) = n$.*

(2) *If $\delta(G) = \lfloor (n-1)/2\rfloor$, then $\chi_i(G) \geqslant \delta(G) + 1$.*

Proof. (1) Since $\delta(G) \geqslant (n+1)/2$, $d_G(u) + d_G(v) \geqslant n+1$ for any two vertices u and v in G. Then u and v have a common neighbor. Hence, $\chi_i(G) = n$.

(2) If $\Delta(G) > \delta(G)$, then $\chi_i(G) \geqslant \Delta(G) \geqslant \delta(G) + 1$. Consider $\Delta(G) = \delta(G) = \lfloor (n-1)/2\rfloor = k$. Suppose $\chi_i(G) = k$ and let $\{V_1, V_2, \ldots, V_k\}$ be the set of color classes of an injective k-coloring of G. If $|V_i| \leqslant 2$ for all i, then $n = \sum_{i=1}^{k} |V_i| \leqslant 2k \leqslant n-1$, a contradiction. Assume that, for some i, V_i contains at least three vertices v_1, v_2, v_3.

Since no two vertices in V_i have a common neighbor, $\Delta(G[V_i]) \leqslant 1$ and hence $n-3 \geqslant |\bigcup_{i=1}^{3} N_G(v_i) \setminus \bigcup_{i=1}^{3}\{v_i\}| \geqslant 2(k-1)+k > n-3$ when $n \geqslant 5$, again a contradiction. ∎

Lemma 5 *Suppose G is a k-regular graph of order $n \geqslant 5$.*

(1) *If $k > n/2$ or $k < (n-2)/2$, then $n+1 \leqslant \chi_i(G) + \chi_i(\overline{G}) \leqslant 2n$.*

(2) *If $k = n/2$ or $(n-2)/2$, then $n \leqslant \chi_i(G) + \chi_i(\overline{G}) \leqslant 2n$.*

Proof. The upper bounds are obvious. Note that, since G is k-regular, $\overline{G}$ is k'-regular, where $k' = n-k-1$.

(1) If $k > n/2$, by (1) of Lemma 4, $\chi_i(G) = n$. Then $\chi_i(G) + \chi_i(\overline{G}) = n + \chi_i(\overline{G}) \geqslant n+1$. If $k < (n-2)/2$, then $k' > n/2$. By (1) of Lemma 4, $\chi_i(\overline{G}) = n$ and then $\chi_i(G) + \chi_i(\overline{G}) = \chi_i(G) + n \geqslant n+1$.

(2) If $k = n/2$, then $k' = (n-2)/2$. By (2) of Lemma 4, $\chi_i(\overline{G}) \geqslant k'+1$ and then $\chi_i(G) + \chi_i(\overline{G}) \geqslant k + k' + 1 = n$. If $k = (n-2)/2$, by (2) of Lemma 4, $\chi_i(G) \geqslant k+1$ and then $\chi_i(G) + \chi_i(\overline{G}) \geqslant k+1+k' = n$. ∎

Theorem 6 *Suppose G is a graph of order $n \geqslant 5$.*

(1) *If $n = 5$ or n is even, then $n \leqslant \chi_i(G) + \chi_i(\overline{G}) \leqslant 2n$.*

(2) *If $n \geqslant 7$ is odd, then $n+1 \leqslant \chi_i(G) + \chi_i(\overline{G}) \leqslant 2n$.*

Proof. The upper bounds are obvious. It is clear that $\chi_i(G) + \chi_i(\overline{G}) \geqslant \Delta(G) + \Delta(\overline{G}) = \Delta(G) - \delta(G) + n - 1 \geqslant n+1$ if $\Delta(G) - \delta(G) \geqslant 2$.

Case 1. $\Delta(G) = \delta(G) = k$.

Then G is k-regular and $\overline{G}$ is k'-regular, where $k' = n-k-1$. If n is even, then Lemma 5 implies $\chi_i(G) + \chi_i(\overline{G}) \geqslant n$. Moreover, suppose that n is odd. If $k \neq (n-1)/2$, $\chi_i(G) + \chi_i(\overline{G}) \geqslant n+1$ by (1) of Lemma 5. If $k = (n-1)/2$, then $k' = (n-1)/2$. By (2) of Lemma 4, $\chi_i(G) + \chi_i(\overline{G}) \geqslant k+1+k'+1 = n+1$.

Case 2. $\Delta(G) - \delta(G) = 1$.

Then $\chi_i(G) + \chi_i(\overline{G}) \geqslant \Delta(G) + \Delta(\overline{G}) \geqslant n$ and (1) is established. Next, let $n \geqslant 7$ be an odd integer. Suppose on the contrary that $\chi_i(G) + \chi_i(\overline{G}) = n$. Then $\chi_i(G) = \Delta(G)$, $\chi_i(\overline{G}) = \Delta(\overline{G})$ and $\Delta(G) + \Delta(\overline{G}) = n$. Without loss of generality, we may assume $\Delta(G) \geqslant \Delta(\overline{G})$. Hence, $\Delta(G) \geqslant n/2$ which implies $\Delta(G) \geqslant (n+1)/2$. If $\Delta(G) > (n+1)/2$, then $\delta(G) = \Delta(G) - 1 \geqslant (n+1)/2$. By (1) of Lemma 4, $\chi_i(G) = n$ and then $\chi_i(G) + \chi_i(\overline{G}) \geqslant n+1$. Suppose $\chi_i(G) = \Delta(G) = (n+1)/2$. Then $\chi_i(\overline{G}) = \Delta(\overline{G}) = (n-1)/2$ and $\delta(\overline{G}) = (n-3)/2$. Let $p = (n-1)/2$ and $\{V_1, V_2, \ldots, V_p\}$ be the set of color classes of an injective p-coloring of $\overline{G}$. Since

$p = (n-1)/2$, V_i contains at least three vertices v_1, v_2, v_3 for some i. Since no two vertices in V_i have a common neighbor, $\Delta(G[V_i]) \leqslant 1$ and hence $n-3 \geqslant |\bigcup_{i=1}^{3} N_G(v_i) \setminus \bigcup_{i=1}^{3}\{v_i\}| \geqslant 2(\delta(\overline{G})-1)+\delta(\overline{G}) = (n-3)+(n-7)/2$. It follows that $n = 7$ and $N_{\overline{G}}(v_1) = \{v_2, v_4\}$, $N_{\overline{G}}(v_2) = \{v_1, v_5\}$ and $N_{\overline{G}}(v_3) = \{v_6, v_7\}$. Then, in G, we have $v_1v_5 \in E(G)$, $v_2v_4 \in E(G)$, and $N_G(v_3) = \{v_1, v_2, v_4, v_5\}$. Therefore, any pair v_i and v_j, $1 \leqslant i < j \leqslant 5$, have a common neighbor. Then $5 \leqslant \chi_i(G) = (n+1)/2 = 4$, a contradiction. Therefore, $\chi_i(G) + \chi_i(\overline{G}) \geqslant n+1$. ∎

Theorem 7 *For any graph G of order $n \geqslant 5$, $n \leqslant \chi_i(G)\chi_i(\overline{G}) \leqslant n^2$.*

Proof. The upper bound is obvious. Let G be a graph of order $n \geqslant 5$. Suppose $\chi_i(G) = p$ and $\chi_i(\overline{G}) = q$. Let f and g be injective p-coloring and q-coloring of G and $\overline{G}$, respectively. Define a mapping $h : V(K_n) \to \{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$ by $h(u) = (f(u), g(u))$ for all $u \in V(K_n)$. If $h(u) \neq h(v)$ for all vertices u and v, then h is an injective pq-coloring of K_n. Hence, $n = \chi_i(K_n) \leqslant pq = \chi_i(G)\chi_i(\overline{G})$. Suppose $h(u) = h(v)$ for some vertices u and v. Without loss of generality, we may assume $uv \in E(G)$. Since $f(u) = f(v)$, $N_G(u) \cap N_G(v) = \emptyset$. Since $g(u) = g(v)$, $x \in N_G(u) \cup N_G(v)$ for all vertices x in G. Then $N_G(u) \cup N_G(v) = V(G)$ and any vertex x in G is adjacent to exact one of u and v. Suppose $d_G(u) = a \geqslant d_G(v) = n-a$. Then $\chi_i(G)\chi_i(\overline{G}) \geqslant d_G(u)d_{\overline{G}}(v) = a(a-1) \geqslant \lceil n/2 \rceil(\lceil n/2 \rceil - 1) \geqslant n$. ∎

Consider the sharpness of the lower bounds. For $n = 5$, $\chi_i(P_5) + \chi_i(\overline{P_5}) = 5$. For $n = 2k \geqslant 6$, $\chi_i(K_{k,k}) + \chi_i(\overline{K_{k,k}}) = k + k = n$. For $n = 2k+1 \geqslant 7$, $\chi_i(K_{k+1,k}) + \chi_i(\overline{K_{k+1,k}}) = k+1+k+1 = n+1$. For $n \neq 2$, $\chi_i(K_n)\chi_i(\overline{K_n}) = n$.

Now consider the sharpness of the upper bounds. Note that $\chi_i(G) = |G|$ if and only if any two distinct vertices in G have a common neighbor. Using this fact, we may see that $\chi_i(G) + \chi_i(\overline{G}) < 2|G|$ if $1 < |G| < 9$. For $k \geqslant 3$, let $n = 3k + t \geqslant 9$, where $t = 0, 1$ or 2. We construct an auxiliary graph G_{3k} as follows. The vertex set of G_{3k} can be partitioned into three cliques $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$, and $Z = \{z_1, z_2, \ldots, z_k\}$ such that $\{x_i, y_i, z_i\}$ forms a clique for all $1 \leqslant i \leqslant k$. We join a new vertex ∞ to all x_i's in G_{3k} to obtain G_{3k+1} and two new vertices ∞_1 and ∞_2 to all x_i's in G_{3k} to obtain G_{3k+2}. It can be verified that any two distinct vertices in G_n and $\overline{G_n}$ have a common neighbor. Hence, $\chi_i(G_n) + \chi_i(\overline{G_n}) = n + n = 2n$ and $\chi_i(G_n)\chi_i(\overline{G_n}) = n^2$.

4 Square chromatic numbers

Since any pair of vertices that are adjacent or distance 2 apart receive distinct colors in a square coloring, every color class of a square coloring must be an independent set.

Theorem 8 *For any graph G of order n, $n+1 \leqslant \chi_{\square}(G) + \chi_{\square}(\overline{G}) \leqslant 2n$, or equivalently, $n+1 \leqslant \chi(G^2) + \chi(\overline{G}^2) \leqslant 2n$.*

Proof. The upper bound is obvious. Suppose G is a graph of order n and $\chi(G^2) = p$. Let $f = (X_1, \ldots, X_a, Y_1, \ldots, Y_b)$ be a square p-coloring of G with $a + b = p$, $|X_i| = 1$ and $|Y_j| \geqslant 2$ for all i and j. If $a = p$, then $a = n$ and $\chi(G^2) + \chi(\overline{G}^2) = n + \chi(\overline{G}^2) \geqslant n+1$. Suppose $a < p$. Since f is a square coloring, each Y_j is an independent set of G and any vertex u in Y_i has at most one neighbor in Y_j for all $i \neq j$. Hence, $uv \in E(\overline{G})$ for some v in Y_j. Then $d_{\overline{G}}(u, v) \leqslant 2$ for all vertices u and v in $\bigcup_{j=1}^{b} Y_j$. Therefore, $\chi(\overline{G}^2) \geqslant n - a \geqslant n - p + 1 = n - \chi(G^2) + 1$, or $\chi(G^2) + \chi(\overline{G}^2) \geqslant n + 1$. ∎

Theorem 9 *For any graph G of order n, $n \leqslant \chi_{\square}(G)\chi_{\square}(\overline{G}) \leqslant n^2$, or equivalently, $n \leqslant \chi(G^2)\chi(\overline{G}^2) \leqslant n^2$.*

Proof. The upper bound is obvious. Since $\chi(G) \leqslant \chi(G^2)$ and $\chi(\overline{G}) \leqslant \chi(\overline{G}^2)$, the lower bound is a consequence of the Nordhaus-Guddam theorem. ∎

Consider the sharpness of the lower bounds. It is clear that $\chi(K_n^2) = n$ and $\chi(\overline{K_n}^2) = 1$. Hence, $\chi(K_n^2) + \chi(\overline{K_n}^2) = n + 1$ and $\chi(K_n^2)\chi(\overline{K_n}^2) = n$.

Now consider the sharpness of the upper bounds. For $2 \leqslant n \leqslant 4$, it is routine to check that $\chi(G^2) + \chi(\overline{G}^2) \leqslant 2n - 1$ if $|G| = n$. For $n \geqslant 5$, we construct a graph F_n of order n as follows. The vertex set of F_n can be partitioned into a 5-cycle $C_5 = x_1x_2x_3x_4x_5x_1$ and an independent set $Y = \{y_1, y_2, \ldots, y_{n-5}\}$ such that each y_i is adjacent to both x_1 and x_3. It can be verified that any two vertices in F_n, or $\overline{F_n}$, are at distance at most 2. Hence, $\chi(F_n^2) + \chi(\overline{F_n}^2) = n + n = 2n$ and $\chi(F_n^2)\chi(\overline{F_n}^2) = n^2$.

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